AST353 Course Review

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Contents

1 Course Review Structure

This course review was written for AST 353 with Professor Kumar during the Fall 2025 semester. It will contain key ideas from each individual day/lecture of the course. There are summaries of each week of the class that will capture the main points. For some in-class practice problems that I found particularly helpful, I will create a solution box and work out the solution.

- **Weekly Sections** contain each week's lectures.
- Each individual day is written out with the contents and formulas that were contained in that lecture.
- **Concept Boxes** have relevant equations or very important pieces of information.
- **Summary Boxes** contain weekly summaries of the class material.

Of course, given that almost all of the content contained in this course review was either sourced directly from Professor Kumar's lectures or lecture materials, I would like to attribute the information in this review to Professor Kumar and acknowledge that his lecture materials for AST353 were used heavily in the creation of this review. This includes the 5 homeworks and 4 different sets of lecture slides available to us.

2 Week 1

2.1 Week 1: Day 1 (*8/26/25*)

During the first day of the course, we covered some of the topics that we'll be discussing as the semester goes on. This inclueldes the composition of the universe and what the different compoments are.

Concept

The composition of the universe by mass/energy by component is:

4% Ordinary matter,   22% Dark matter,   74% Dark energy

Ordinary matter is also known as baryons and (other than some light elements like H, He, and Li) all elements were produced in the cores and of stars which were dispersed around systems when the stars end their lives.

We then discussed more about subatomic physics and introcued the "new periodic table". Ordinary matter is made up of quarks and electrons, with there being several different kinds of quarks. Quarks have different masses, charges, spins, and names. There are "up", "charm", "top", "down", "strange", and "bottom" quarks. Quarks combine to create different important buildling blocks of matter. Protons have a fractional charge because they are made up of three quarks.

Concept

Subatomic particles:

- Protons are made of two up quarks and one down quark
- Neutrons are made of two down and one up quark.
- Most subatomic particles can be produced in particle accelerators.

Neutrinos are small subatomic particles that don't interact much with matter at all because they interact using the weak atomic force. Analogous to electric charges are strong interactions and their assignments.

2.2 Week 1: Day 2 (*8/28/25*)

There are four basic 'forces' or interactions between matter that we can observe.

Concept

The four basic interactions are:

Gravity, Electromagnetism, Strong force, Weak interaction

An asterisk here is that gravity isn't really a force when considered through the lens of general relativity. This is not very relevant when broadly defining the important interactions.

An extremely important concept we were introduced to today was metrics and the curvature of spacetime. Just like how the surface of a sphere has curvature, the fabric of spacetime itself can be curved or flat, and this influences geometry in the space. Metrics are a way of determining the distance between two nearby points in a space. Our universe is a special case in which $k = 0$, meaning the space part of our universe is flat. The use of metrics is also how we know that the distance between points is increasing. Another way of saying this is that the scale factor othe universe, $a^2(t)$ is increasing with time; "The yardstick for time is changing with time".

Concept

Some Important Metrics:

$$\text{Cartesian metric:} \quad ds^2 = dx^2 + dy^2 + dz^2$$

$$\text{A spherical coordinate metric in a flat space:} \quad ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

$$\text{Metric for surface of a sphere:} \quad ds^2 = R^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

$$\text{Metric for the universe:} \quad -ds^2 = -c^2 dt^2 + a^2(t)\left(\frac{1}{1-kr^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right)$$

$$\text{Hyperbolic surface:} \quad ds^2 = \left[\frac{1}{1+r^2}\right] dr^2 + r^2 d\phi^2$$

$$\text{Minkowski space:} \quad x^2 + y^2 - c^2t^2 = -1$$

$$\text{Minkowski metric:} \quad -ds^2 = -c^2 dt^2 + dx^2 + dy^2$$

Each metric is defined using different kinds of coordinates. Cartesian coordinates use a regular x, y, z coordinate system, while spherical coordinates are maded up of r, θ, ϕ which represent the radius outward from the origin, the polar angle, and the azimuthal angle to another point. Knowing this coordinate information for two points (and subtracting to get the difference) allows each metric to give distances between the points.

The final week 1 topic we covered was a practical application of relativity and spacetime effects. Namely, we talked about GPS satellite networks. GPS satellites have to account for relativistic effects because the mass of the clocks closer to the Earth tick faster than those of satellites above in orbit. Satellites carry atomic clocks with accuracy on the order of nanoseconds, and they transmit their own measured time as well as their location information to a receiver. Without accounting for these effects, GPS would lead to very large errors that compound over time, making it nearly useless after a few days of these relativistic errors. To accurately allow a reciever to determine their position, they need signals from at least 4 satellites: 3 to triangulate distance and one to make an accurate measurement of time.

The timing equation for these satellites is:

$$|R_{\text{you}} - R_{\text{satellite}}| = c(t_{\text{you}} - t_{\text{satellite}})$$

2.3 Week 1: Summary ()

Week 1 introduced some of the broad themes of the class, such as the matter/energy composition of the universe, subatomic particles and the basics of how atoms are built, metrics and curved spacetime, as well as relativistic effects and practical consideration of them. The next topic we will discuss is the cosmic microwave background as well as cosmic distances and measurement.

3 Week 2

3.1 Week 2: Day 1 (*9/2/25*)

To begin week 2, we discussed the Hubble constant (H_0). The Hubble constant represents the expansion rate of the universe, and was first calculated by Edwin Hubble. As we know from week 1, the expansion rate of the universe is acclerating, so the units of the Hublle constant is velocity over distance ($(km/s)/Mpc$). One major topic in modern astrophysics is Hubble tension, which is the difference in value for the Hubble constant between two different measurement methods, one which uses “standard candles” (Cepheid variable stars and supernovae), and the other which uses a “standard ruler” (the Cosmic Microwave Background).

Standard Candles

Standard Candles are events or observations for which we know the luminosity. These are very useful when used in conjunction when considering flux dropoff, because we can calculate the distance between the detector and event.

Concept

Distance from luminosity

$$d = \sqrt{\frac{L}{4\pi F}}$$

Where L is luminosity, d is distance, F is apparent brightness.

Concept

Hubble constant from supernovae (doppler shift)

$$H = v/d$$

$$H^2 = H_0^2[\Omega_{m0}(1+z)^3 + \Omega_{A0}]$$

Where H_0 is the initial Hubble constant, Ω_{m0} is the density of dark matter and baryons, and Ω_{A0} is the density of dark energy.

One of the most widely used standard candles is Cepheid variable stars, which are stars that dim and brighten periodically, and have an extremely consistent variation between this period and their luminosity. From measuring this dimming period, their luminosities become known, and we can reliably find distances between ourselves and Cepheid variables that are 30 Mpc away.

The other most important type of candle is a type Ia supernova. These supernovae occur when white dwarf stars with binary companions accrete matter onto themselves from their partner, and eventually reach a breakdown point at which the dwarf star gravitationally collapses. Given that we know at which mass these supernovae occur, all type Ia supernovae have about the same luminosities. These events can be seen from up to 3000 Mpc away.

Standard Rulers

The Cosmic Microwave Background (CMB) is the thermal imprint of the Big Bang which can be measured to give us fundamental and important information about the universe. One of the conclusions we can draw from the CMB is through baryon acoustic oscillations, which are ripples in the distribution of matter from the very early universe, which are imprinted in the thermal power spectrum of the CMB.

At a redshift of $z = 1100$, the universe underwent recombination, which is when neutral atoms were formed out of photons and plasmas after the expansion of the universe allowed the temperature to cool enough. These oscillations can roughly be thought of as sound waves or density waves through the universe that originated from the Big Bang. From the wavelengths of these oscillations, we can construct a “standard ruler” which allows us to measure length scales by comparing wavelengths to angular sizes we observe, which is useful if we know the speed at which these density perturbations travelled.

3.2 Week 2: Day 2 (*9/4/25*)

Gravitational Waves

Gravitational radiation, or gravitational waves, are distortions in the curvature of spacetime which propagates at the speed of light (or the speed of causality). Distances between points in space oscillate when there are gravitational waves present. They are emitted whenever there is a change in the configuration in mass or energy over time. While gravitational waves propagate whenever this happens, they are usually so weak that we can not detect them at all. When massive, dense objects such as neutron stars and black holes merge, this creates immense amounts of energy, which causes gravitational waves strong enough for us to measure.

The leading way to measure gravitational waves is through interferometry. Observatories like LIGO have laser interferometers that measure laser phase to determine path length and detect any changes on the order of 10^{-17} cm, which is even smaller than the hydrogen nucleus.

LIGO also detected the collision of two neutron stars, which confirmed theoretical calculations that about a third of all elements in the periodic table are produced in neutron star merges.

Black Hole Photography + Quasars

The supermassive black hole in the center of the Milky Way, M87, was photographed using the Event Horizon Telescope. The telescope is spread between 7 different millimeter telescopes across the globe, which gives impressive resolution. The "ring" around the picture of M87 is a "licht echo" in which the light that circles the black hole is amplified due to gravitational effects.

Stars that wander too close to black holes are sheared apart due to tidal forces caused by the immense gravity of black holes. This causes the material from the star to accrete around the black hole, and either accrete around the black hole or fall inside. This causes a quasar, an extremely energetic stream of particles, to jet out of the top radially of the black hole.

Gamma Ray + Fast Radio Bursts

When super massive stars with $\sim 35M_\odot$ end their lives, they die in a supernova. We can detect these extremely energetic bursts as gamma rays, which last a few seconds. As a rule of thumb, we observe a few of these a day.

Fast Radio Bursts (FRBs), discovered in 2007 by the Parkes 64m radio telescope, are signals around 1.5 GHz that have recently been found. Neutron stars are theorized to produce these due to crystal shear. When these waves travel through plasma in the universe, higher frequency waves reach us first, and we are able to use FRBs to create a map.

Concept

Properties of Fast Radio Bursts (FRBs):

Duration: $\delta t = 5ms$

Flux: $30 \pm 10\ Jy$

$V_{EM}(v)$ **increases with v**

$DM = \int n_e\ dl = 375\ cm^{-3}\ pc$

$\delta t(\lambda) \propto \lambda^{4.4}$ consistent with pulse broadening due to interstellar medium turbulence.

Aside on life in the universe

To finish off week 2, we talked about the possibility that we are not alone in the universe. Exoplanet detections from Kepler show a significant amount of planets being found with periods and radii very comparable to the Earth. Additionally, we can analyze the amount of exoplanets that fulfill the habitability condition (i.e being the correct distance from a star to be able to hold liquid water) and identify candidate planets from our observations of other star systems. Of course, we don't have any definitive answer (yet) about whether there is other life in the universe.

3.3 Week 2: Summary ()

Week 2 was spent doing a broad overview of several important topics in astrophysics. Among these were the Hubble constant as well as standard rulers, and Hubble tension. Other topics were gravitaional waves, black holes (photography and interaction with stars), supernovae, gamma ray bursts, and fast radio bursts. Finally, we had a brief discussion on the search for extraterrestrial life in regards to exoplanets we observe.

4 Week 3

4.1 Week 3: Day 1 (*9/9/25*)

We began to discuss the big bang and the differences between popular depictions of it from the generally accepted scientific concensus on the big bang. Importantly, the big bang was NOT a bomb in any kind of pre-existing empty 3D space. Einstein's theories instead lead us to believe that the big bang was an event in which space itself expanded indredibly rapidly. Gamma ray bursts are the biggest explosions in our universe since the big bang.

Dark Matter intro

Some good ways to think about the distinction between dark matter and dark energy is that dark matter has a regular gravitational attraction with matter and tends to clump up on galaxy scales. Meanwhile, dark energy has repulsive gravity and, as far as we can tell, is uniformly distributed across the universe.

So, then we can just use Newton's/Kepler's laws to detect dark matter, by measuring its gravitational impact.

Concept

Classical laws useful for dark matter detection:

$$\frac{m_1 v^2}{R} = \frac{GMm_1}{r^2}$$

$$M = \frac{Rv^2}{G}$$

Where m_1 is a smaller mass orbiting a larger mass, R is the radius of orbit, M is the mass of the larger object, G is the gravitational constant, and v is the orbital velocity.

The further out one goes from a galaxy's center radially, the more dark matter is detectable using these classical laws. By mass, dark matter starts to dominate at round 10 *kpc* radially outward from a galaxy's center.

Concept

Dark Matter's properties:

- Can not have an electric charge or interact using electromagnetic forces.
- Has no nuclear charge.
- Has mass and interacts with normal matter gravitationally.

We will discuss evidence for dark matter much more in depth during later lectures, but some of the most compelling evidence for the existence of dark matter includes the rotational velocities of spiral galaxies, velocities of galaxy clusters, x-ray luminosities and gas temperatures in galaxy clusters, as well as gravitational lensing. Some indirect evidence includes the growth of density fluctuations and the formation of early stars and galaxies, as well as the CMB.

Galaxy-cluster Collisions

Galaxy clusters, which are made up of an immense amount of galaxies and are some of the largest structures in the universe, sometimes collide with each other and we can roughly model this collision as two spheres, the interiors made up of hot gas and the exteriors made of collisionless dark matter, passing through each other. The two hot spheres of gas merge after the collision and separate from the dark matter, leaving dark matter shells.

4.2 Week 3: Day 2 (*9/11/25*)

More on Dark Matter

The universe has about 5x more dark matter than it does ordinary matter, and by measuring gravitational effects, we can determine that dark matter is very nearly uniformly distributed across the universe.

Concept

Example problem: Calculate the amount of dark matter in this room! Given:

- Dark matter is uniformly distributed in the galaxy (a sphere of about $r = 10kpc$)
- The total stellar mass of the galaxy is about $t \times 10^10 M_\odot = 1.5 \times 10^{44} g$
- There is 10X as much dark matter in the universe as there is ordinary matter
- The galaxy's observable radius is about $\sim 10kpc$
- Mass of dark matter particles is 10 GeV

If dark matter mass is 10 times the stellar mass:

$$M_{\text{DM}} = 10 \times (1.5 \times 10^{44}\,\text{g}) = 1.5 \times 10^{45}\,\text{g}$$

And the radius is:

$$R = 10\,\text{kpc} = 10 \times (3.086 \times 10^{21}\,\text{cm}) = 3.086 \times 10^{22}\,\text{cm}$$

We then need the densities:

$$\rho_{\text{DM}} = \frac{M_{\text{DM}}}{V_{\text{gal}}} = \frac{1.5 \times 10^{45}}{1.23 \times 10^{68}} \simeq 1.2 \times 10^{-23}\,\text{g}\,\text{cm}^{-3}$$

Dark matter particle mass:

$$m_{\text{DM}} = 10\,\text{GeV} = 10 \times 1.78 \times 10^{-24}\,\text{g} = 1.78 \times 10^{-23}\,\text{g}$$

We get the volume:

$$V_{\text{gal}} = \frac{4\pi}{3} R^3 = \frac{4\pi}{3}(3.086 \times 10^{22})^3 \simeq 1.23 \times 10^{68}\,\text{cm}^3$$

Then number density is:

$$n_{\text{DM}} = \frac{\rho_{\text{DM}}}{m_{\text{DM}}} = \frac{1.2 \times 10^{-23}}{1.78 \times 10^{-23}} \simeq 0.67\ \text{particles/cm}^3$$

Direct Detection of Dark Matter

There is much work being done to try and directly detect dark matter, with any potential direct interactions with matter instead of gravitational detection. A plausible candidate for dark matter are Weakly Interacting Massic Particles (WIMPs), which pass between matter without interacting strongly. Many experiments are being carried out to detect potential dark matter interactions. The hope is that a WIMP collides with a particle nucleus, and it is believed that the interaction would be weaker than the weak interaction. There are cryogenic dark matter detectors which use stable elements such as liquid xenon to search for WIMPs that pass through the detector. Liquid xenon must be extremely cold, at about $T \sim 0.001\ K$.

Hypothetically, WIMPs that pass through the detector would produce flashes of light, but none have been detected so far. Experiments and detectors such as this serve to give us an upper limit for how large the particle effective size could be, and in 2022 the ZEPLIN team found that the largest possible size for dark matter particles are $30GeV = 6 \times 10^{-48} cm^2$.

4.3 Week 3: Summary ()

During week 3, we discussed dark matter in depth. This included gravitational detection of dark matter, some points of evidence for its existence, and efforts underway to detect dark matter candidates.

5 Week 4

5.1 Week 4: Day 1 (*9/16/25*)

Dark Energy

Ordinary atoms are called baryons. Other than baryons and dark matter, the universe has dark energy which is related to mass by Einstein's famous mass energy relation.

Dark energy makes up the majority of the content of the universe by mass/energy, and is responsible for the repulsive gravity that expands the universe. Dark matter is more smoothly distributed than ordinary matter in the universe, but is relatively clumped due to density variations. However, dark energy is completely uniformly distributed throughout the universe.

Given that our universe is completely flat in terms of its curvature, and dark + ordinary matter do not account for the energy content needed for this to be true, we say that there is dark energy which makes up this deficit. Dark energy's density remains constant as space expands, while ordinary + dark matter have their density decrease.

Observational Evidence for Dark Energy

The greatest observational evidence for dark energy is given by measurements of type Ia supernovae, which were introduced in week 2. Supernovae were found to be dimmer than expected from galaxies that had multiple supernovae observed. Their distances were calculated under the assumption that the expansion of the universe was constant, but it was clear after these standard candle measurements that the expansion was accelerating. We attribute this accelerating expansion to dark energy, and its repulsive gravity is what expands space.

Universe Models

By imagining different models of universes that contain different densities of dark energy, dark matter, and ordinary matter, we can examine how the behavior of the universe is affected by this balance.

Concept

Einstein de Sitter Universe A universe with only baryons and dark matter ($\Omega_{M0} = 1$). Geometry is curved in this model of universe.

Concept

de Sitter Model Universe The solution to Einstein's equations that only contains the cosmological constant and expands exponentially.

Observational evidence of type Ia supernovae disagree with both the above universe models. The expansion rate implied by supernovae is only explained with a mix of densities such that

$$\Omega_t ot = \Omega_{Baryon} + \Omega_{DM} + \Omega_{DE}$$

This density of 1, the critical density, is a required condition for our geometry to be Euclidean (flat). In fact, we say that there's exactly enough dark energy in the universe to make the universe exactly flat and reach the critical point.

Density Component Domination

As the universe has expanded, its density has crossed from being dominated by ordinary and dark matter to being dominated by dark energy. We can deduce this by calculating the scale factor of the universe when it is dominated by dark matter and ordinary matter, and recognizing that it is different from dark energy. Given that dark energy's density remains the same as the universe expands, we think that dark energy caused the expansion to start accelerating $6Gyr$ ago, when its density became dominant, but dark energy has always been present.

Dark Energy Candidates

One well-known candidate to explain dark energy is quantum fluctuations of fields within vacuum, but actual calculations yield a value of energy too large by a factor of $\sim 10^{120}$ to explain acceleration.

The cosmological constant encodes that every space intrinsically has an electromagnetic field inside it. Due to this, quantum fluctuations are always present everywhere; you can not have a totally empty space. Everything has quantum fluctuations which contribute to the energy associated with space when we calculate this. It's way larger than it should be.

5.2 Week 4: Day 2 (*9/18/25*)

CMB Power Spectrum

Taking a power spectrum of data gives us a way to examine its periodicity. When taking a power spectrum of the CMB, we are essentially examining the periodic variations in its temperature and looking at what patterns are strongest. Doing this gives us information about the composition and geometry of the universe.

Recall from earlier lectures that summing all the normal and dark matter with the mass equivalent of dark energy tells us that the universe has just the right density to be flat. The era of cosmic inflation, which will be discussed more in later lectures, stressed space and wiped out any curvature in pre-existing space, making it flat.

An important point to note is that the critical density of the universe is not constant, but decreases exactly with the current density, in 'lockstep.'

Curved Spacetime

While Newton's laws of gravity were used for centuries, there were some problems and observations for which they failed to make the correct prediction. Our theories of gravity would be overhauled by Einstein, when he published his theory of general relativity. He proposed that gravity is not a force, but is rather the curvature of spacetime due to mass.

According to Einstein, planets and other bodies that are in orbits are actually simply moving on the geodesic, the shortest possible line, in curved spacetime (the earlier lectures on metrics are relevant here).

Arguments for Curvature

Our ancestors developed some ingenious arguments to determine that the Earth is a sphere. One such argument used the shadows cast during lunar eclipses to determine that the Earth's shadow was round, like the 2D projection of a sphere. The earliest records of a spherical Earth being accepted stretch back to 400 B.C.

To determine whether the 3D space we live in is curved, we must create definitions for lines in space. We define a 'straight line' to be the shortest or longest line between two points. These arguments will be developed further using metrics next week.

5.3 Week 4: Summary ()

Week 4 introduced lots of information about dark energy and the Cosmic Microwave Background. We learned about how we know dark energy exists, its role in expanding the universe, as well as how different universes with different distributions of matter/energy behave. Using CMB observations, we also examined how to construct a power spectrum. We finished the week by poerperly introducing Einstein's theory of relativity and how it differs from Newton's, as well as what observational evidence or arguments could be made to verify this.

6 Week 5

6.1 Week 5: Day 1 (*9/23/25*)

We continued this week beginning from the question introduced in the previous lecture: How can we determine straight lines (geodesics) and curvature in general spaces? Recall from previous lectures that a metric determines the distance between nearby points. Metrics operate on different spaces, so we can examine what these different metrics look like.

Concept

Flat-space metrics:

$$\text{Rectangular: } ds^2 = dx^2 + dy^2 + dz^2$$

$$\text{Rectangular w/ conversions: } ds^2 = f_x^2 dx^2 + f_y^2 dy^2 + f_z^2 dz^2$$

$$\text{Spherical: } ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

These are metrics for flat 3D spaces.

Metrics for flat spaces differ from those for spaces with curvature, and we can introduced some metrics that operate on curved spaces.

Concept

Metric for a positively curved 2D space (surface of a sphere), with finite volume.

$$ds^2 = R^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Metric for a hyperbolic surface (negative curvature)

$$ds^2 = \left[\frac{1+z^2}{1-z^2}\right] d\rho^2 + \rho^2 d\phi^2$$

The metric of the universe, the FRW metric, is a special case in which there is no curvature ($k = 0$).

Concept

FRW Metric

$$-ds^2 = -c^2dt^2 + a^2(t)\left[\frac{1}{1-kr^2}dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2\right]$$

6.2 Week 5: Day 2 (*9/25/25*)

More on Metrics

Metrics can be used to calculate geodesics, the shortest or longest lines between two points (usually we are interested in the shortest). Not only can we find the geodesics, but a metric also gives us the ability to calculate the curvature of a space.

Concept

Definition: Geodesics are straight lines" between two points:
In an n-dimensional space, a curve is described by coordinates $x_i(T)$ as functions of a parameter T.

We can generalize our definition of a metric to be some function of the coordinates we choose for a space:

$$ds^2 = P_1(x_1, \ldots, x_n)dx_1^2 + \cdots + P_n(x_1, \ldots, x_n)dx_n^2$$

Then the distance along a curve between parameters T_1 and T_2 is:

$$S = \int_{T_1}^{T_2} \left[P_1(x)\dot{x}_1^2 + \cdots + P_n(x)\dot{x}_n^2\right]^{1/2} dT, \qquad \dot{x}_i = \frac{dx_i}{dT}$$

Additionally, we also have the important constraing that the distribution of matter/energy in our energy is exactlly the critical density. This is the same as saying:

$$P_n(\dots) = 1 \text{ for flat Cartesian space. For curved spaces } P_n(\dots) \neq 1.$$

Concept

Example problem: Calculate a geodesic on a sphere! The metric is given as

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2$$

so

$$P_1(\theta, \phi) = R^2, \qquad P_2(\theta, \phi) = R^2 \sin^2\theta.$$

Define some parametric curve:

$$\theta(T) = \pi^2, \qquad \phi(T) = \sin T$$

Differentiate:

$$d\theta = 2\pi \, dT, \qquad d\phi = \cos T \, dT$$

Substitute into sphere metric:

$$ds^2 = 4R^2\pi^2 dT^2 + R^2(\sin^2\theta)(\cos^2 T)dT^2,$$

Euler-Lagrange Equations

Generally, we can use the Euler-Lagrange equations (ELEs) to extremize metrics, and find the geodesics in a particular space. Treat the metric as some function F:

$$F(x_i, \dot{x}_i) = P_1(x)\dot{x}_1^2 + \cdots + P_n(x)\dot{x}_n^2$$

And we know then that the geodesic follows from the ELEs:

$$\frac{\partial F}{\partial x_i} - \frac{d}{dT}\left(\frac{\partial F}{\partial \dot{x}_i}\right) = 0, \qquad i = 1, \dots, n.$$

6.3 Week 5: Summary ()

This week developed much of the machinery we will use to determine the shortest path between points in arbitrary spaces given that we know their metrics. We examined the importance and effects of curvature on these calculations, and did some example problems. Additionally, the Euler Lagrange equations were introduced as a method of using variational calculus to extremize integrals and find geodesics.

7 Week 6

7.1 Week 6: Day 1 (*9/30/25*)

Continuing from our discussion of metrics and the ELEs from last week, we dove deeper into variational calculus to apply our machinery to finding geodesics.

For some family of curves, we can parameterize each member of the family by distance or arc length s. For a general metric of the form

$$ds^2 = P_1(x_1, \dots, x_n)\dot{x}_1^2 + \cdots + P_n(x_1, \dots, x_n)\dot{x}_n^2,$$

the functional we want to extremize is

$$F = P_1(x)\dot{x}_1^2 + \cdots + P_n(x)\dot{x}_n^2$$

For flat 2D space, the metric becomes

$$F = \dot{x}^2 + \dot{y}^2$$

Applying the ELEs:

$$\frac{d}{d\tau}(2\dot{x}) = 0, \qquad \frac{d}{d\tau}(2\dot{y}) = 0,$$

so

$$x = a\tau + b, \qquad y = c\tau + d$$

Which gives us the intuitive answer that straight lines in this space have constant velocity. Choosing two endpoints like

$$A(0,0,0), \qquad B(3,4,2),$$

immediately gives us the geodesic which is linear.

Concept

Example: Find geodesics on the surface of a torus! Through geometric analysis, we can see that the metric is given by:

$$ds^2 = r^2 d\theta^2 + (R + r\cos\theta)^2 d\phi^2$$

The ELE for ϕ gives a conserved quantity:

$$(R + r\cos\theta)^2 \dot{\phi} = k$$

In the special case $\dot{\phi} = 0$ at some point, the constant is $k = 0$, and the motion is purely in the θ direction. Different initial points and conditions can give very interesting curves which might not be what one expects to see when thinking about "straight lines."

Concept

Example: Find geodesics in a 2D curved space! We have the metric

$$ds^2 = R^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

The function to extremize then looks like

$$F = R^2(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2)$$

The ELEs for ϕ gives us a conserved quantity

$$\sin^2\theta\, \dot{\phi} = a$$

The equation for θ becomes

$$\ddot{\theta} = -a^2 \frac{\cos\theta}{\sin^3\theta}.$$

Integrate:

$$\dot{\theta}^2 = -a^2 \frac{1}{\sin^2\theta} + b.$$

By rotating our coordinate system so that one point lies at the "north pole" where $\theta = 0$, ϕ becomes constant along the geodesic. This trick simplifies the problem and allows us to see that geodesics in this space are great circles, which are circles along the equator of the sphere.

The Equivalence Principle

As we now know, gravitational paths taken by orbiting bodies are simply 4D geodesics, with spacial dimensions and time being the relevant coordinates. An important observation about gravity is that there is no way to distinguish between uniform acceleration and uniform gravity. This fact is known as the equivalence principle. A famous thought experiment around the equivalence principle is that if you were in a rocket with no windows or way to observe the outside and felt a force pulling you downwards, it would be impossible to determine whether that force is due to the rocket remaining still on the surface of a planet exerting gravity or due to the rocket accelerating.

Concept

Example: Equivalence Principle in an elevator
Consider a photon in an accelerating elevator, where the acceleration is a. If the photon initially has frequency f and is shot upward, we can denote the time it takes for the photon to travel a distance h:

$$t = h/c$$

The elevator velocity is then $v = at = ah/c$ We can find the Doppler shifted photon frequency:

$$f_{new} = f(1 - v/c) = f(1 - ah/c^2)$$

This is known as gravitational redshift

Gravitational Redshift

As shown in the elevator example, gravitational redshift is the decrease in photon frequency and energy caused by gravity pulling it downward.

Concept

Gravitational Redshift: $\frac{v^2}{2c^2}$ when gravity is weak ($v_{esc} << c$)

Gravity also causes time dilation, where frames that experience more gravity experience time moving faster due to the distortion of spacetime with mass.

7.2 Week 6: Day 2 (*10/2/25*)

We consider a spherically symmetric expanding universe that has the metric:

$$-ds^2 = -c^2dt^2 + a^2(t)\Big(dr^2 + r^2d\Omega^2\Big)$$

For photons, we know that their frame demands that:

$$0 = -c^2 dt^2 + a^2(t) dr^2 \quad \rightarrow \quad \frac{dr}{dt} = \frac{c}{a(t)}$$

If we have two points that differ by δr in terms of coordinates, we can use the metric to determine the actual physical distance between the points. Additionally, it's important to notice that the comoving coordinates are systems that shift as distances between the coordinates change, which are especially useful to avoid problems of the coordinates we specify to different points not encoding information about their distances without metrics.

In a matter dominated universe, the scale factor is

$$a(t) \propto t^{2/3}.$$

Then,

$$\frac{dr}{dt} = c\,t^{-2/3}.$$

Integrating,

$$r(t) = 3c\,t^{1/3}\lambda,$$

which rearranges to

$$r \propto \frac{\lambda}{t}.$$

7.3 Week 6: Summary ()

This week had lots of practice with using metrics to find geodesics for a variety of situations, as wlel as discussing the vital equivalence principle which allowed us to examine gravitational redshift and some of the consequences of Einstein's theory of relativity.

8 Week 7

8.1 Week 7: Day 1 (*10/7/25*)

Gravity as Curved Spacetime

Matter and photons follow geodesics in spacetime, not because a "force" pulls on them, but because the geometry of space itself is curved. Neither space or time are rigid.

Uniform acceleration is also the same as uniform gravity, as we know from the equivalence principle discussed last week.

We now consider two different types of mass. Inertial mass and gravitational mass are distinct concepts abstractly, but they actually turn out to be the same.

To illustrate this, we consider a photon in an accelerating elevator, similar to the setup in which we examined gravitational redshift. Its frequency is Doppler shifted:

$$f_{\text{new}} = f(1 - v/c) = f\left(1 - \frac{ah}{c^2}\right)$$

With elevator velocity $v = at = ah/c$. By the equivalence principle, this is the same as a gravitational field causing energy loss in the photon.

Then we can apply this to Earth's gravity:

$$f_{\text{new}} = f\left(1 - \frac{gh}{c^2}\right)$$

For a typical height such as 3000 cm and $f \approx 3 \times 1014\ Hz$:

$$f_{\text{new}} = f\left(1 - \frac{980}{c^2} \times (3 \times 10^{14})\right) = \text{a very small number}$$

Concept

Because frequency is the inverse of time, gravitational redshift implies time runs more slowly deeper in a gravitational field.

When launching two photons in the same upward direction in our setup at different times, frequency decreases, so photon height lines that are supposed to be parallel actually increase in distance. Another way to notice curved spacetime to notice curved spacetime is by measureing the difference in velocity between photons that are launched at different points in the field.

8.2 Week 7: Day 2 (*10/9/25*)

We built more on the equivalence principle, beginning with the idea that photon energy decreases with ascending height.

Consequences of the Equivalence Principle

An interesting implication that arises from the equivalence principle is that time flows slower at the surface of the Earth than it does above the surface. Taking this thought to its logical extreme, we can imagine what this would be like for a black hole. At the horizon of a black whole, our time would pass so much faster than the rest of the universe that we would be able to see the entire universe evolve and would essentially live eternally (if we somehow survived).

As a thought experiment, a ship moving at $v = 0.999c$ experiences significant time

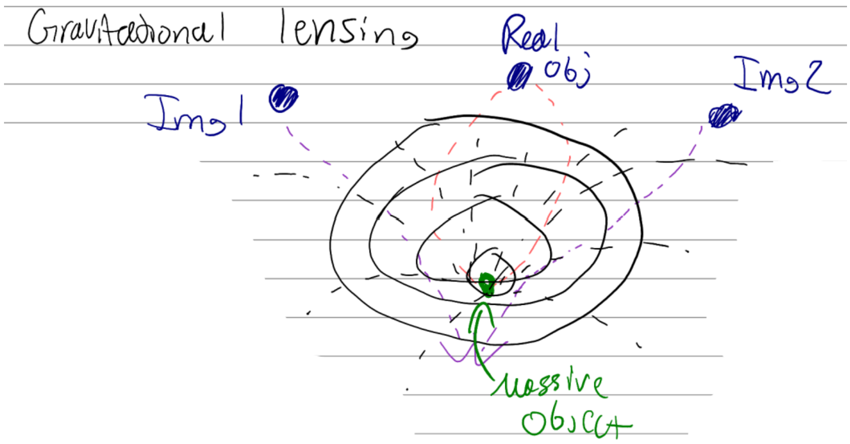

Figure 1: A diagram of microlensing. Light is bent by a massive object like a lens, creating multiple images.

dilation compared to observers on Earth and passengers would live longer compared to people on Earth.

Section Summary

Both gravitational potential and velocity slow the passage of time.

Geodesics and Light

Another consequence of Einstein's relativity is that massive object curve spacetime so that light rays passing near it follow curved geodesics. Netwon's laws could not predict the paths that light would take correctly due to gravity, but the relativity approach does make correct predictions.

Gravitational Microlensing

When making astronomical observations, a single background object can appear as multiple images. This is known as gravitational microlensing, and occurs when light form one object is bent around a massive object between the light and observer like a lens, displaying multiple images. A diagram of this is shown in figure 1.

Microlensing also leads to Einstein rings, which are circular rings of light that appear when a gravitational lens bends incoming light around itself in such a way that the image is distorted around into a circle.

The Cloverleaf Quasar is one of the most famous examples of this phenomena, in which 4 images of a single quasar was detected in the X-ray.

Eclipse Experiments

The experiment which made Einstein a household name overnight was led by Sir Arthur Eddington in 1919, and it involved observing the stars near the sun during a total solar eclipse. These observations were carried out in several parts of the world, and they determined that the light from the stars had indeed been shifted by the gravity of the sun, and they were able to use this observational data to confirm Einstein's hypothesis.

Precession of the Perihilion of Mercury

One of the most well-known experiments that was a confirmation of relativity was the precession of perhilion of Mercury. Mercury's orbit was historically an anomaly, due to its orbit having anomalies that Newton's theory of gravity could not properly account for.

Mercury's orbit precesses by about $43''$ per century beyond what Newtonian gravity can explain. General relativity accounts for the additional precession almost exactly, with the additional 2° per century.

$$\Delta\phi = v_{\text{esc}}^2/c^2$$

at Mercury's orbital radius.

Relativity and GPS

Expanding on our previous discussion about GPS and realtivistic effects, recall that a GPS receiver must detect signals from at least four satellites. The position is determined from the system:

$$|R_{\text{you}} - R_{\text{sat}}| = c(t_{\text{you}} - t_{\text{sat}})$$

the unknowns are the three space coordinates and the time offset.

GPS satellites orbit Earth at a speed

$$v = \sqrt{\frac{GM_E}{R}}$$

and this speed causes time dilation. But the satellites are also higher in Earth's gravitational potential, where clocks run faster, so there are multiple effects to account for.

The combined correction to the onboard clock is:

$$t_s = t_E\left(1 - \frac{GM_E}{Rc^2}\right)\left(1 - \frac{GM_E}{R_E c^2}\right)$$

which has to be applied for GPS to remain accurate.

Without these corrections, the accumulated error in one day would be:

$$\text{57 microseconds} \quad \rightarrow \quad \text{GPS off by } \sim 10 \text{ miles}$$

8.3 Week 7: Summary ()

This week focused heavily on relativity and observables that confirm Einstein's theories. Among these were the precession of Mercury, gravitational lensing, and solar eclipse observations. We also looked further into the relativistic effects that GPS satellites must account for now that we have the machinery to do so.

9 Week 8

9.1 Week 8: Day 1 (*10/14/25*)

Distance Scales in Cosmology

There are a few different ways astronomers define distance in an expanding universe. Local systems in which expansion changes the distance between things negligibly or not at all have simple, fixed distances. But because galaxies far away from each other have to take into account the expansion of the universe, we introduce comoving coordinates.

Luminosity Distance

Luminosity distance is a method of defining distance that relies on standard candles. By taking the equation for flux as a function of luminosity and distance, we can rearrange to get an expression for distance:

$$F = \frac{L}{4\pi d_L^2}$$

so

$$d_L = \sqrt{\frac{L}{4\pi F}}$$

Luminosity distance grows not only because an object is farther away, but it also increases because photon energy is redshifted and the arrival rate of photons slows by a factor of $1 + z$.

We can also use an associated standard ruler that has a true known size to convert luminosity distance to physical distance. This standard ruler is angular size in the sky: $\delta\theta$:

$$d_A = \frac{L}{\delta\theta}$$

In an expanding universe:

$$L = a(t)\, r\, \delta\theta \quad \to \quad d_A = a(t)r = \frac{r}{1+z}.$$

$$d_L = (1+z)^2 d_A.$$

Relating Comoving Distance and Luminosity Distance

For a 3D space with no curvature ($k = 0$), a light source emits photons in a shell spread over $4\pi(a_0 r)^2$ where a_0 is the scale factor of the universe today. The energy of each photon is smaller by a factor of $a_0/a = 1 + z$ because of expansion. So then, radiation emitted over a time δt is received over a time that is $\delta t(a_0/a) = (1 + z)\delta t$. This allows us to relate luminosity distance (d_L) and comoving distance (r) using the definition of flux:

$$F = \frac{L}{4\pi(1+z)^2 r^2} \rightarrow d_L = (1+z)r$$

Angular Diameter distance

We can find the distance between us and a source in the sky by measuring the angular diameter it occupies in the sky, using the FRW metric with $d\phi = 0$. Across the object in the sky, its length is then $\delta\ell = a(t)r\delta\theta$ given that we observe its angle to be $\delta\theta$ in the sky, and the nominal distance between us (r) is affected by the scale factor of the universe $a(t)$. Then angular diameter distance (d_A) is:

$$d_A = \frac{\delta\ell}{\delta\theta} = a(t)\ r$$

or

$$d_A = \frac{r}{1+z}$$

Which is related ti luminosity distance by a factor of $(1 + z)^2$.

Horizon Distance

Horizon distance is another type of distance commonly used by astronomers, and it is defined as the distance that light has travelled since the big bang. This relies on comoving coordinates, which are present in the FRW metric:

$$-ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2\right]$$

The comoving coordinates are (r, θ, ϕ) and are unchanged even as the universe expands. For a flat universe ($k = 0$):

$$\text{proper distance} = a(t)r$$

If we consider photons that begin travelling at a time t:

$$\int_{t_e}^{t_0} \frac{c\,dt}{a(t)} = r,$$

Now considering a second photon that leaves at a time δt later, the relative redshift due to the expansion of the universe becomes

$$1 + z = \frac{a(t_0)}{a(t)}$$

We also know how the scale factor for a matter dominated universe behaves, with

$$\frac{a}{a_0} = (t/t_0)^{2/3}$$

This gives us an extremely useful expression for obtaining the age of the universe at a particular redshift:

Concept

Age of the universe at redshift z: $t = t_0(1+z)^{3/2}$

Then the horizon distance can be found with

$$d_H = a(t)\int_0^t \frac{c\,dt'}{a(t')}$$

and for a matter-dominated universe $a(t) \propto t^{2/3}$, so:

$$d_H = 3ct$$

9.2 Week 8: Day 2 (*10/16/25*)

Geometry and Expansion of the Universe

Some fundamental assumptions we have used thus far when examining the properties of the unvierse are that:

1. The universe is statistically the same everywhere. Properties like the number of galaxies per unit volume as well as the sizes and shapes of galaxies are statistically the same in every part of the universe at any given time.

2. On large scales, the universe is homogeneous and isotropic, meaning it is the same and is identical in every direction and there are no structures on the largest scales.

Knowing whether a universe has these properties is important because it lets us cut down the amount of possible geometris that a universe can have. Namely, we know that a universe that has these properties has to be:

- spatially closed (positive curvature),
- flat (zero curvature), or
- open (negative curvature).

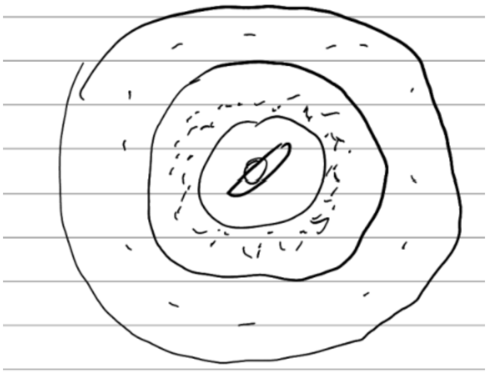

Figure 2: A diagram of an isotropic but inhomogeneous universe. There is a central density cluster, and density decreases in every direction outward from the center.

Does homogeneous imply isotropic?

An interesting thought experiment we considered was asking, "Is a universe that is homogenous also have to be isotropic?" Universes indeed can have directionality and repeating structure. An easy example to consider is a universe with an intrinsic magnetic field that points in one particular direction rather than in a random direction like our universe.

An isotropic but inhomogeneous universe could take the form of a universe where there is a central dense region that grows less dense radially outward. A diagram of this is shown in figure 2.

Expansion of Universe in the far future

Luckily, for our universe which is both homogeneous and isotropic, the geometry can be found by simply measuring mean density. We can calculate the curvature of our universe by doing tensor calculus, which is the formally correct method, or we could do a mock proof using the Newtonian method which is simpler and gets most of the way to the correct conclusions.

To follow the Newtonian method, we consider a test galaxy at distance R from the center of a large sphere of mass M. The energy of this galaxy is then

$$E = \frac{1}{2}mv^2 - \frac{GMm}{R}$$

A few different cases emerge depending on the sign of the energy.

- $E < 0$: The galaxy moves back down into the center of the mass cluster because its kinetic energy does not win against the potential.

- $E > 0$: An open solution in which the galaxy escapes outward forever.

- $E = 0$: The case in which the galaxy stays in its current position.

These different cases correspong to the three types of universes discussed previously, and this Newtonian method is much less effort than the formal proof.

9.3 Week 8: Summary ()

Throughout this week, we discussed different astronomical distance measurements, how they relate with eachother, and their coordinates. We also discussed the consequences of our universe being isotropic and homogeneous, and how the geometry/density of the universe can be characterized.

10 Week 9

10.1 Week 9: Day 1 (*10/21/25*)

Expansion rate of the Universe

We've now established over several lectures that the density of a universe is extremely important to iots dynamics. But the form of the existing energy/mass is also of vital importance. For example, the expansion rate of the universe when most of the energy/mass is in the form of ordinary and dark matter is different compared with the expansion rate when the universe is dominated by radiation. Now we'll derive the expansion rates (scale factors) of three cases and see how they compare.

We call the scale factor of the universe $R(t)$, and use a test model of a large spherical region of the universe with radius R and recall the Friedmann equations:

Concept

Friedmann Equation 1

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\rho - \frac{k}{R^2}$$

Friedmann Equation 2

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p/c^2)$$

Knowing the Hubble constant is the expansion rate of the universe, $H = \dot{R}/R$. Then the critical density of the universe (ρ_{crit}) can be found in terms of the Hubble constant by asserting that a totally flat universe is at the critical density.

Concept

Critical Mean Density of the Universe

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$

So: If $k > 0$, then $\rho > \rho_{crit}$ and the universe will collapse in time. If $k < 0$, then $\rho < \rho_{crit}$ and the universe will expand forever.

We can get a number for the critical density of our universes given a Hubble constant. This becomes

$$\rho_c = \frac{3H^2}{8\pi G} \approx 8.9 \times 10^{-30}\,\text{g cm}^{-3}$$

Now that we have information about the expansion rate of the universe and its dependence on energy/matter density, we can determine what the expansion dynamics are when this density is in different forms.

Case 1: Universe dominated by matter ($p \approx 0$)

In this case, the total mass inside our test sphere remains constant, so using the energy equation:

$$v^2 \propto R^{-1} \qquad v = \dot{R} \qquad R\dot{R} \propto R^{-1/2}$$

Integrate:

$$R(t) \propto t^{2/3}$$

Which tells us how the universe expands in the ordinary/dark matter dominated case.

Section Summary
Matter and dark matter dominated universe expands with $R \propto t^{2/3}$.

Case 2: Universe dominated by radiation ($p = \frac{1}{3}\rho c^2$)

Knowing the energy that is present in radiation (Planck's constant multiplied by the frequency), we can say that radiation's energy density is proportional to radiation pressure. We know that for the pressure of radiation: $p \propto V^{-4/3} \propto R^{-4}$.

Due to wavelength stretching, which would decrease frequency:

$$v^2 \propto R^{-2} \qquad R\dot{R} \propto R^{-1} \qquad R \propto t^{1/2}$$

Section Summary
Radiation dominated universes expand as $R \propto t^{1/2}$.

Case 3: Universe dominated by dark energy ($p = -\rho c^2$)

We know that dark energy has constant energy density no matter how much space expands. So,

$$\frac{\ddot{R}}{R} = +\frac{4\pi G}{3}\rho_\Lambda,$$

which is positive, and this explains why dark energy related expansion accelerates:

$$R(t) \propto e^{Ht}$$

Section Summary

Dark energy dominated universe expands exponentially with: $R \propto e^{Ht}$.

10.2 Week 9: Day 2 (*10/23/25*)

Radiation from the Big Bang

The Cosmic Microwave Background (CMB) is the radiation left over from the univere's birth, and while it has been mentioned and we have briefly discussed what information we can obtain from it, we now take a deeper dive into its properties and existence.

The universe was much hotter at earlier times, especially during its first few hundred thousand years of life. Hot gas cools with expansion with $T \propto V^{-1/3}$. Knowing that the CMB temperature today is 2.73 K, we can determine background temperature at an earlier redshift with

$$T = 2.73(1+z)\,\text{K}$$

We can determine times/redshifts at which the universe had certain temperatures:

$$T \sim 10^9\,\text{K at } t = 1 \text{ sec} \qquad T \sim 3000\,\text{K at } z \approx 1100 \qquad t(z) = \frac{t_{pres}}{(1+z)^{3/2}}$$

The redshift $z = 1100$, or $370,000$ years into the universe's life, corresponds to an extremely important time in the life of the universe: recombination.

Thermal History of the Universe

Now we know that different eras of the universe were dominated by different types of energy. At the earliest times, the universe was dominated by radiation, and we can work backwards form the current CMB temperature to determine what it must have been earlier in the universe's life.

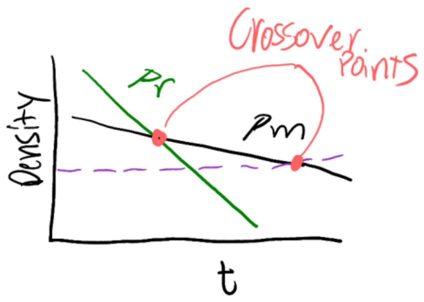

Figure 3: A diagram of an isotropic but inhomogeneous universe. There is a central density cluster, and density decreases in every direction outward from the center.

With $T_0 = 2.73K$ being the CMB temperature and the mass density in radiation then following from $\rho_{r0} = a_{BB}T_0^4/c^2$ (also knowing that the radiation constant $a_{BB} = 7.57 \times 10{-}15\ ergcm^{-3}K^{-4}$), we can use the fact that the energy density of a radiation dominated universe depends on the scale factor as $\rho_r \propto a^{-4}$ to compare with the matter dominated case. Knowing that the current matter density of the universe is $\rho_{m0} = 2.8 \times 10^{-30} g\ cm^{-3}$, we can determine that a radiation dominated universe was a factor of $\rho_{m0}/\rho_{r0} = 6 \times 10^3$ smaller than it is now. A qualitative graph and crossover points of density between radiation, matter, and dark energy domination are shown in figure 3.

The figure also illustrates how the dark energy density at early times, and also somewhat during the matter dominated regime, was very small compared to radiation and matter density. Given that the energy density of radiation was almost exactly the critical density in the early universe, we can find out more about the radiation temperature as well.

We already know that for radiation, $\rho = f_d \times \rho_0$ where f_d is the number of different particles that constitute the radiation. Then, we can equate this to the critical density:

$$\rho = \rho_c = \frac{3H^2}{8\pi G}$$

Using the definition of the Hubble constant $H = \dot{a}/a$ as well as the time dependence of the scale factor $a \propto t^{1/2}$, we can find that:

$$T = \left(\frac{3c^2}{32\pi G f_d a_{BB} t^2}\right)^{1/4} \approx 1.5 \times 10^{10} K t^{-1/2}$$

This allows us to find the temperature of the universe as a function of time after the big bang, but only when the universe's energy density was still dominated by radiation.

Concept

The universe, due to expansion, has cooled from an initial temperature of $10^9\ K$ from 1 second after the big bang to $2.7\ K$ now.

Recombination Intro

This is the era in which hydrogen unionized, and before this point the universe was opaqua due to the properties of this medium. We know that when $T \geq 3000$, hydrogen behaves as $H \to p^+ + e^-$, which is how we know at what temperature recombination occured.

10.3 Week 9: Summary ()

This week had lots of information about the dynamics of the universe in regards to how the expansion rate differs when the energy density is contained in different mediums, and then painted a picture of how the newborn universe behaved. This included talking about Big Bang nucleosynthesis (BBN) and how this era affects the chemical composition of our universe today, as well as how we can use the CMB to determine the temperatures, and then times/redhsifts, at which important events like recombination happened.

11 Week 10

11.1 Week 10: Day 1 (*10/28/25*)

Recombination Continued

As we began talking about last week, the reason we can observe the universe when it was younger is because it was transparent at that time. This was, however, not always the case. H_2 gas ionized at a temperature of $3,000$ K, and it acts like a 'fog' that obscures our attempts to observe back to times in which hydrogen in the universe as ionized. We now know how temperature changes with redshift, so we can calculate the redshift required to see ionized hydrogen gas.

Concept

$$T(z) = T_0(1+z) \qquad \text{and} \qquad t/t_0 = (1+z)^{-3/2}$$

$$(1+z) = 3 \times 10^3 \quad \to \quad z \approx 1100$$

$$t \approx 370{,}000 \text{ yrs}$$

The reason this ionized hydrogen acts like a fog can be explained by considering the interactions that electrons have with light. Electrons have an effective cross section which scatters light and causes the universe's medium to become opaque to us. This effect is called Thomson scattering.

Concept

Thomson scattering cross section:

$$\sigma_T = 6.65 \times 10^{-25}\,\text{cm}^2.$$

Using the cross section, we can consider the mean free path of photon interactions with electrons. Mean free path is a statistical model in which we can think about the average path length that an object takes before interacting with some other object, informed by densities and velocities of the objects.

Before recombination, photon mean free path was extremely small:

$$\ell_{\text{mfp}} = \frac{1}{\sigma_T n_e}$$

which is much smaller than the size of the universe. This argument explains why light scattered all over the place when these free electrons pre-recombination were present.

Spherical Fourier Analysis

A quick aside on Fourier analysis is useful before discussing CMB data processing in detail. Fourier analysis essentially decomposes a function into a sum of sines and cosines, and examines the periodicity in the function. The usual form for a Fourier transform is

$$\tilde{X}(f) = \int_{-\infty}^{\infty} X(t)\ \sin ft\ dt$$

But we have to add an additional dimension to this to obtain something that's more useful for CMB data. Because we examine the CMB spherically in the sky, we can use Foureier analysis to find the spherical harmonics and power spectrum of the data. Then the power spectrum for this data is taken with:

$$C_{L,m} = \int T(\theta,\phi)\ \gamma_{L,m}(\theta,\phi)\ d\theta d\phi$$

In a sense, this is analogous to electron orbital configurations because due to the parameter choices in this integral and the quantum numbers in electron orbitals. The two parameters L, m describe the harmonic order that is present. We say that L is the degree and m is the order. Then we have two requirements for these paramter choices:

$$L \in \mathbb{Z}^+$$

$$-L \leq m \leq L$$

Projecting these shapes onto data lets us analyze the spherical periodicity of the data.

Sound Horizon

We know that sound waves are essentially perturbations of density within a medium, and the fluctuations in CMB temperature can be thought of as similar density perturbations in the hot early universe. These perturbations would have travelled at the speed of sound of the medium, and the peak of the CMB power spectrum, the largest periodic contribution of the Fourier modes, is determined by the sound horizon. The speed of sound in the early universe is

$$c_s = \frac{c}{\sqrt{3}}$$

And we can find the sound horizon as

$$d_s(t) = a(t)\int_0^t \frac{c_s\, dt'}{a(t')}$$

Then for a matter dominated universe, this ends up being

$$a(t) \propto t^{2/3}, \quad d_s(t) =\approx \sqrt{3}ct$$

We can find the angular size at the current epoch with

$$\theta_s = \frac{d_s(t_{\text{rec}})}{r(t_0)}.$$

So at recombination, $z = 1100$, $\delta\theta_{sh} = 1$.

11.2 Week 10: Day 2 (*10/30/25*)

A possible resolution of Hubble Tension

Recall that the Hubble constant has different values when derived from the CMB vs when using measurements of standard candles. This difference that is larger than the uncertainty in the measurements is known as Hubble tension.

Concept

Measurements of the Hubble constant
CMB: $H_0 = 67.4 \pm 0.5$
Standard Candles: $H_0 = 74 \pm 2$

A proposal that seeks to rectify Hubble tension makes the case that the universe expanded rapidly because of some different form of dark energy (not the same dark energy expanding our universe today) slightly before recombination. This decreases the size of the sound horizon and suggests that the CMB underestimates the Hubble constant, which would explain the deviation between the values. Given that the sound horizon is an experimental measurement (and is thus fixed), the sound horizon can then be decreased if $a(t)$ was larger than expected before recombination time:

$$dsh = a(t_R)\int_0^{t_R}\frac{c_s\, dt'}{a(t')},\ \delta\theta_{CMB} = \frac{d_{sh}}{D_A}$$

Particle Creation in the Early Universe

One important thing to note is that applying Einstein's energy mass relation to this intense radiation that was present at the beginning of the universe shows us how light interacts to become matter. Given that average photon energies are $E = 3k_BT$, when $E > mc^2$ for radiation interactions, particle and anti-particle pairs are created.

Considering an extremely high temperature, almost uniform density universe packed with radiation, which is how the newborn universe very likely was, we can start to look at how matter evolved as the universe began starting from this state.

Knowing this information, there are two important events tied to the universe's temperature that we can look at in detail.

Baryogenesis and Big Bang Nucleosynthesis

Baryogenesis occured about 10^{-6} seconds after the big bang, and was the temperature threshold at which baryons could form out of the primordial 'soup' of the quark gluon plasma. This temperature is the maximum temperature at which neutrons start forming, and these neutrons will decay into electrons and protons. This decay and combination scheme looks like this:

$$p + e^- \Longleftrightarrow n + \bar{\nu}_e$$
$$n + e^+ \Longleftrightarrow p + \bar{\nu}_e$$
$$n \Longleftrightarrow p + e^- + \bar{\nu}_e$$

An important moment to consider is when the reaction rate for the above reacions drops below the expansion rate of the universe, because after this point in time, the ratio of protons to neutrons is fixed. This is known as freezeout, and it occured at about 10 minutes after the big bang after accounting for the fact that free neutrons decay on their own.

This starts the epoch of Big Bang nucleosynthesis (BBN) in which most of the light elements that exist in the unvierse are made. About 24% of atoms by mass in our universe are helium-4. With our current models of stellar nucleosynthesis, we can not account for the vast majority of helium within our universe. BBN accounts for this helium by giving us a synthesis chain that happens in the first few minutes after the big bang. First, ^{3}He is synthesized with these interactions:

$$d + p \to \ ^3\text{He} + \gamma \tag{11.1}$$
$$d + d \to \ ^3\text{He} + n \tag{11.2}$$

Some unstable forms of ^{3}He also decay:

$$d + d \to \ ^3\text{H} + p \tag{11.3}$$
$$^3\text{He} + n \to \ ^3\text{H} + p \tag{11.4}$$

Now everything that's required to synthesize the helium that's common in our universe today, ^{4} is present. The following interactions create it:

$$d + d \to \ ^4\text{H} + \gamma \tag{11.5}$$
$$d +^3 \text{He} \to \ ^4\text{H} + p \tag{11.6}$$
$$d +^3 \text{H} \to \ ^4\text{H} + n \tag{11.7}$$

During BBN, any elments heavier than beryllium-7 were not synthesized. So all elements that are heavier than this were forged in the cores of stars.

There are two key ideas that are important to keep in mind from this discussion:

Concept

Particle number density:

$$h_P = g_P \left(\frac{m_p k_B T}{2\pi h^2}\right)^{3/2} \exp\left\{\left(-\frac{m_p c^2}{k_B T}\right)\right\}$$

Concept

Freezeout: $p + e \leftrightarrow n$ eventually when interaction rates fall below the expansion rate of the universe, the reactants and products 'decouple' and the ratio is frozen out.

11.3 Week 10: Summary ()

There were three major ideas this week:

The era before recombination, in which hydrogen gas existed in an ionized state, was opaque to use due to the way the free electrons diffracted light.

We can study the CMB's patterns and periodicity using Fourier analysis techniques and draw lots of powerful conclusions about the universe using these. This includes the temperature of the early universe, creating a standard ruler with the angular size of the sound horizon, etc.

Right after the big bang, we can study how temperature decreased, and how this brought on the creation and combination of matter in the universe in Big Bang Nucleosynthesis, which is responsible for all the chemistry and elements we observe in the universe today.

12 Week 11

12.1 Week 11: Day 21 (*11/4/25*)

Star and Galaxy formation

Stars and galaxies did not form until $\sim 10^8$–10^9 years after the big bang. These structures were created by density fluctuations in the universe (overdensities) eventually growing and collapsing from their own gravity. Overdensities grow as the universe expands, and we can examine how they grow using models similar to how we determined how the unvierse expands when mass density is dominated by different things.

First we can consider a sphere of gas that has a radius R, with internal pressure P and density ρ

Pressure force per unit volume:

$$\frac{P}{R}$$

Gravitational force per unit volume:

$$\frac{GM\rho}{R^2}$$

Equilibrium with gravity requires

$$\frac{GM}{R} = C_s^2$$

where C_s is sound speed

Collapse happens when

$$V > C_s$$

The free fall time is shorter than the sound crossing time:

$$t_{ff} = \frac{R}{V} = \frac{3}{8\pi\sqrt{G_p}} < t_s$$

Other ways to quantify the collapse criteria are when

$$\frac{R}{V} < \frac{R}{\sqrt{2}} C_s, t_{ff} < t_s$$

These criteria are called Jean's instability critera. Jean's length is the minimum radius of this sphere for which it becomes unstable and will gravitationally collapse. Jean's mass is the mass at which this happens (related by density).

Concept

Instability conditions:

$$V^2 < \frac{GM}{R}, \qquad t_{\text{ff}} < t_s.$$

Jeans length:

$$R_J = \frac{3C_s}{\sqrt{8\pi G\rho}}.$$

Jeans mass:

$$M_J = \frac{4\pi}{3}R_J^3\rho.$$

Now knowing the criteria overdensities have to fulfill to collapse gravitationally, we can examine what happens in an expanding universe and how this affects overdensities.

Density Perturbations in an expanding Universe

If the spherical overdensity we considered previously was in an expanding universe and has density $\rho = \rho_0(1+\delta)$, we can use the second Friedmann equation to write:

$$\ddot{R}_0 = -\frac{GM}{R_0^2}, \ddot{R}_0 + \ddot{R}_1 = -\frac{GM}{(R_0 + R_1)}^2$$

$$\rightarrow R_1 = -\frac{2R_0R_1}{R_0}$$

Where R_0 is the initial radius of the sphere and R_1 is the additional length added by the universe's expansion. Then the mass:

$$M = \frac{4\pi\rho_0R_0^3}{3} = \frac{4\pi(\rho_0+\rho_1)(R_0+R_1)^3}{3}$$

Which leads to:

$$R_1 = -R_0\frac{\rho_1}{3\rho_0} = -\frac{R_0\delta}{3}$$

$$\rightarrow R_1 = -\frac{1}{3}(R_0\delta + 2R_0\delta + R_0\delta)$$

where $\delta = \rho_1/\rho_0$ is the fractional change in density.

This allows us to finally obtain a differential equation that describes how the density changes:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_0\delta = 0$$

Which can also be written

$$\ddot{\delta} + 2H\dot{\delta} = \frac{3}{2}H^2\Omega_m\delta.$$

The solutions of this equation depend on expressions for density, so they will be different for different eras of the universe that have mass density in different forms.

Radiation Dominated Case

Radiation domination specifies that the matter density is negligible and the Hubble constant is $H = 1/(2t)$. Plugging this into the perturbation equation, we get:

$$\delta(t) = C_1 t + C_2 t^{-1}$$

Which is a differential equation that has the general solution of $\delta(t) = c_1 + c_2 \ln(t)$. Notably this shows that overdensity growth is quite slow in the radiation dominated case.

Matter Dominated Case

If we specify that the dark energy density and radiation energy density are negligible, $\Omega_A = 0, \Omega_R = 0$, then the Hubble constant becomes $H = 2/(3t)$. The differential equation becomes:

$$\ddot{\delta} + \frac{4\delta}{9t} - \frac{2}{3t^2}\delta = 0$$

This equation has a power law solution,

$$\delta(t) = b_1/t + b_2 t^{2/3} \propto t^{2/3}$$

So perturbations grow proportionally to the scale factor of the universe in this case.

Dark Energy Dominated Case

For datk energy, we can take $\Omega_m \approx 0$, so the Hubble constant is actually constant. Then, let this constant be H_A. The solution for the resulting differential equation gives us

Dark-energy dominated:

$$\delta(t) = a_1 + a_2 e^{-2H_A t}$$

Which essentially means that perturbation growth freezes entirely.

12.2 Week 11: Day 2 (*11/6/25*)

Inflation and the Multiverse

A potential explanation for what existed before the big bang is that there was a "quantum foam" that was suddenly injected with a very large amount of dark energy due to quantum fluctuations. This causes the space to expand rapidly from a tiny region into the size of a universe. To determine why this is a valid theory for the origin of our universe, we have to examine two "problems" in astrophysics that left astronomers puzzled for decades.

The Flatness Problem

As we discussed in earlier parts of the course, the geometry of our universe is extremely flat ($\rho \sim \rho_{crit}$). The possibility of this happening is so low that it can not be explicable by chance alone. To demonstrate this, Professor Kumar had a wonderful demo in which he invited a student to balance a pen on its tip unsuccesfully. But surprisingly, he was able to do it (through the use of a magnet). The point of this demo was to show that when something extremely unlikely happens, we don't just attribute it to chance, we try to see what could have caused it (in this case the magnet).

Flatness would require an extremely special initial condition, and we can use the first Friedmann equation to see that:

$$\frac{\dot{R}^2}{R^2} - \frac{8\pi}{3}G\rho = -\frac{k}{R^2}$$

This leads us to

$$(1-\Omega) = -\frac{k}{R^2H^2}$$

Substitute in $(1-\Omega_0) = -\frac{k}{R_0^2H_0^2}$:

$$\frac{1-\Omega}{1-\Omega_0} = (1+z)^2\frac{H_0^2}{H^2}$$

If we're just considering a matter dominated universe, we know that the Hublle constant is $H = H_0(1+z)^{3/2} \rightarrow \frac{1-\Omega}{1-\Omega_0}\frac{1}{1+z}$. WE can work backward from here to see that when the universe was the Planck temperature, $T = 10^{19}\ GeV$, the initial density satisfied $(1-\Omega) \approx 10^{-62}$. This is such an astronomically low difference between the critical density, that like the pen balancing on its tip, we can not just trust that it was chance!

Inflation, as we'll discuss more in detail, proposes that the reason space is so flat is that it was stretched an extreme amount in an initial period of dark energy expanding the universe dozens of orders of magnitude.

The Horizon Problem

When viewing the CMB, we observe multiple areas of the universe that are statistically identical, but actually have no causal connection with each other (meaning they are accelerating away from each other faster than the speed of light, so no causation can occur between them). Recalling angular diameter distance, we can calculate the horizon distance:

$$3ct_0^{2/3}\left(t_0^{1/3} - t^{1/3}\right)$$

Horizon distance:

$$d_H = a(t)\int_0^t \frac{c\, dt'}{a(t')} = 3ct$$

The angular size at the current epoch is $\delta\theta_H = \frac{1}{(1+z)^{1/2}-1}$.

Then this angular size and causal separation can also be explained by a period of intense early cosmic inflation. If two very similar close-by regions has the space between them expanded by an extreme amount, these two similar regions can become disconnect from each other causally. So in summay,

Concept

- Start with a universe of arbitrary curvature and many small, causally connected regions.
- Put this universe through a period of intense exponential inflation.
- After inflation, regions originally in contact become separated from each other to an extreme degree.
- Curvature becomes almost totally flat, which solves the flatness problem.
- The observable universe looks mostly homogeneous because it originated from a small initial patch with similar properties.

Background Fields

Fields are objects that prescribe a scalar value (or sometimes vector or tensor) to every point they encompass. Some background fields that exist in our universe are

- Density field $\rho(\vec{r}, t)$
- Velocity field $\vec{v}(\vec{r}, t)$
- EM Fields $\vec{E}(\vec{r}, t), \vec{B}(\vec{r}, t)$

A more complicated field that exists in our universe is a quantum field. Quantum fields assign an operator, instead of a scalar or vector, to each point. As an example: Quantum Electrodynamics (QED) uses what's known as a 4-vector potential:

$$(\phi, \vec{A})$$

It's also useful to note that in some fields of physics, natural units are used, which assign certain fundamental constants to be 1.

Concept

Some common natural units

$$c = 1, \quad G = 1, \quad \hbar = 1$$

The reason we introduce quantum fields is because the simplest field capable of causing inflation is the scalar quantum field. The Lagrangian density of such a scalar field is given by

$$\mathcal{L} = \sum_{\mu=0}^{3} \left[\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - V(\phi) \right].$$

Taking the extrema of the action integral of this allows us to study the dynamics of a field like this.

12.3 Week 11: Summary ()

Section Summary

This week saw us address two key points: how do density perturbations grow as the universe grows and how does this affect the creation of structure in the universe? And, what are the flatness and horizon problems and what is the resolution to them?
We used Newtonian test models to discuss the growth of density perturbations and introduced the era of cosmic inflation to address the horizon/flatness problems, as well as what could have come before the big bang.

13 Week 12

13.1 Week 12: Day 1 (*11/11/25*)

The current size of the observable universe is known to be:

$$R_{\text{obs}} \sim 10^{29} \text{ cm}$$

But the size of the observable universe is not constant at different redshifts. So, at redshift z, the size would become

$$R_{\text{obs}}(1+z), \qquad a(z) = a_0/(1+z)$$

We can also find out the size of the observable universe at the Planck temperature, $T \sim 10^{16}\,\text{GeV} \sim 10^{29}\,\text{K}$:

$$R_{\text{obs}}\left(\frac{10^{10}\,\text{eV}}{10^{16}\,\text{GeV}}\right) \sim 10\,\text{cm}$$

So when the universe was the Planck temperature, causal connections only existed on the order of centimeters. This is another point in favor of the cosmic inflation theory, because it shows that causal connections grew drastically (a solution to the horizon problem).

Recalling the Lagrangian density of a scalar quantum field introduced in the previous lecture:

$$\mathcal{L} = \sum_{\mu=0}^{3}\left[\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + V(\phi)\right]$$

The coordinates (x^0, x^1, x^2, x^3) are the same as the ct coordinates, (ct, x, y, z).

Using the FRW metric:

$$ds^2 = -c^2dt^2 + a(t)^2(dx^2 + dy^2 + dz^2),$$

the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{a(t)^2}\left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right)\right] + V(\phi)$$

The left portion is essentially the "kinetic energy of the field." Plugging this into the ELEs:

$$\frac{\partial^2\phi}{\partial t^2} + 3H\frac{\partial\phi}{\partial t} - \nabla^2\phi + \frac{\partial V}{\partial\phi} = 0.$$

If we're considering a homoogenous field, then $(\nabla^2\phi = 0)$, which simplifies our expression to:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$$

Concept

A quadratic potential

$$V(\phi) = \frac{1}{2}m^2\phi^2$$

Has ϕ oscillating around the minimum potential like a well.

We can evaluate the energy momentum tensor at a time and position t and x to get:
Density:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{1}{2a^2}\sum_i (\partial_i \phi)^2$$

Pressure:

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) - \frac{1}{6a^2}\sum_i (\partial_i \phi)^2$$

We can see that when $\phi_t^2 << V(\phi) - \phi_i \approx 0$:

$$p \approx \frac{1}{2}\dot{\phi}^2 - V(\phi) \qquad \rho \approx \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

And

$$p \approx -V(\phi), \qquad \rho \approx V(\phi).$$

Using the first Friedmann equation:

$$H^2 = \frac{8\pi\rho}{3} \qquad (G = 1)$$

Putting everything together and using the second Friedmann Equation 2:

$$R(t) = R_i \exp\left(\int_0^t H(t')\, dt'\right)$$

with

$$H = \left(\frac{4\pi}{3}\right)^{1/2} m\phi(t)$$

The density is 'sub-Planckian,' and this implies (as we asserted earlier) that some kind of dark energy associated with an inflation field.

Concept

Some form of dark-energy–like behavior in the scalar field is probably responsible for inflation and implied eternal inflation.

13.2 Week 12: Day 2 (*11/13/25*)

Black Holes

This week we started looking in-depth on black holes, which are not just fictional objects but are very real and have extreme properties. A black hole is an object that has a density so high that light can not escape its gravitational pull. Einstein's theory of general relativity was instrumental in predicting and describing black holes due to Newton's laws not being able to account for the behavior of systems with extreme gravity.

Concept

The radius at which light can not escape a black hole is known as the Schwarzschild radius: $R_s = 2GM/c^2$
This ends up being a few kilometers for a black hole with the mass of the sun, and several millimeters for a black hole with the mass of the Earth.

At the center of black holes are 'singularities,' which are points that (according to the math) have infinite densities. Given that we can not observe the black hole inside its event horizon, we can not be sure what happens inside a black hole.

Black Hole Information

When an object collapses into a black hole, the only information we can recover from it are its electrical charge, which is usually extremely small, its mass, and angular momentum. Other properties of the black hole are totally invariant to its origin.

Black Hole Evidence

The main observational evidence for the existence of black holes comes from our measurements of their gravitational effects. This includes gravitational waves that stem from particularly violent events involving black holes, powerful radiation (quasars) that happen as gas and stars fall into black holes, and also stars moving around unseen objects at high velocities.

13.3 Week 12: Summary ()

This week, we finished our discussion of the period of cosmic inflation by examining the dynamics of scalar quantum fields that would be able to exponentially expand the universe. We then began covering black holes, starting from their basic properties and then introducing some of the observational methods that allos astronomers to directly see that they exist.

14 Week 13

14.1 Week 13: Day 1 (*11/18/25*)

There are two main categories of black holes that we see: stellar mass black holes and supermassive black holes.

Concept

Stellar mass black holes: $\sim 5 - 20M_{sun}$ with $\sim 10^7$ in our galaxy.
Supermassive black holes: $\sim 10^6 - 10^9 M_{sun}$ and they appear in galactic centers.

Black Hole Binaries

When black holes are in binary systems, we can observe their gravitational influence on their partner. Usually, a telltale sign that we've found a black hole in a binary system is when the black hole's binary companion has material stolen and accreted around the black hole. This gas then gets superheated by the black hole's gravity and starts glowing in the X-ray, which we can observe.

Gas that falls into a black hole in this way is heated an extreme amount, to temperatures of $\sim 10^7 K$, which radiates in the X-ray. This is the most efficient way of producing energy in the universe, converting about 20% of the rest mass of the gas into energy.

Cygnus X-1 was the first such X-ray source, and it was discovered in the 1970s. It is the first and most well-known stellar-mass binary black hole candidate.

Concept

Finding binary black hole candidates: If an X-ray emitter is observed in a binary system where there is only one visible binary partner, use Kepler's laws to determine the mass of the unseen companion. There are only two possibilities for what the unseen companion is, it is either a neutron star or a black hole because only those two objects have the gravity required to heat gas to emit X-ray. If the found mass is more than a neutron star can be ($\sim 3M_{sun}$), then the companion must be a stellar mass black hole.

Black Hole Gravitational Waves

Black hole merger events, in which two black holes collide and merge into one, cause an extreme amount of energy to be released and are some of the most violent events in the universe. This process releases enough energy in the form of gravitational waves for us to be able to measure using laser interferometry on Earth. LIGO, a laser interferometry observatory that has a precision of less than the width of a proton, detected gravitational waves from a black hole merger in 2015.

The Galactic Center

Now that we have some methods to determine stellar mass black hole candidates, how do we observe supermassive black holes? Unfortunately, we can't see the galactic center easily at visual wavelengths due to the high density of gas and dust clouds which obscure much of what we want to see. But both the X-ray and infrared wavelengths allow us to better make out what's going on. Then, we can observe the orbital velocities of stars in the galactic center. Genzel, Ghez, and Penrose won the Nobel prize in physics in 2020 for their work observing stars orbiting the center of our galaxy, in which they saw orbital velocities as high as $12,000\ km/sec$, implying the existance of an extremely massive object.

Using Kepler's laws and the measurements of Genzel, Ghez, and Penrose, we know:

$$d = 950\ AU, P = 15.2\ yr$$
$$\rightarrow M = \frac{d^3}{P^2} = 3.7 \times 10^6\ M_{sun}$$

Which is surefire evidence for the existence of a supermassive black hole in the center of our galaxy.

Black Hole Photography

The famous image of the M87 black hole shows light circling around the black hole, caught a little outside its event horizon. Some of the light escapes and hits our detectors, which is what allows to form the image. The image of M87, which is about 16 Mpc away, was taken using the Event Horizon telescopes, which are 7 telescopes spread out across the globe to provide a staggering angular resolution.

The light from the M87 photograph can also be polarized to show the geometry of the black hole's magnetic field. This shows field lines moving across the light.

Black Hole spacetime

We can find the metric for a black hole, starting from Einstein's equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ and solving it for a point mass. This metric is known as the Schwarzschild metric.

Concept

Schwarzschild metric $-c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{rc^2}\right)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$

This metric simplifies to the Minkowski metric as $r \rightarrow \infty$, which makes intuitive sense because spacetime stops being affected by the black hole as the distance from the black hole increases.

It's also important to note that plugging in the Schwarzschild radius into this metric will produce a mathematical singularity, but this point physically denotes the point at which one can not escape the pull of a black hole.

Particle 4-momentum

We can encode information about a particle's coordinates in a 4-momentum vector, which is given with:

$$p^\mu = m\left(c\,\frac{dt}{d\tau},\ \frac{dx}{d\tau},\ \frac{dy}{d\tau},\ \frac{dz}{d\tau}\right) \equiv m\,(ct,\ \dot{x},\ \dot{y},\ \dot{z})$$

$$v^i = \left(\frac{dx}{dt},\ \frac{dy}{dt},\ \frac{dz}{dt}\right)$$

where τ is the proper time in the particle comoving frame, and v^i is the regular 3-dimensional velocity where time is measured from some stationary observer. The reason the 4-momentum is useful is because we can use it with our metrics to find the paths of the particles.

Concept

Minkowski metric for flat space-time

$$-c^2 d\tau^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

We can used these in tandem to see how the particle's movement will evolve in a particular space that we know the metric for (in this case the Minkwoski metric):

$$\left(\frac{d\tau}{dt}\right)^2 = \left[1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dy}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dz}{dt}\right)^2\right] = \left[1 - \frac{v^2}{c^2}\right]$$

This simplifies into

$$\frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \equiv \gamma$$

The t-component of the 4-momentum gives us

$$p^t = mc$$

which is a restatement of Einstein's mass energy relation just differing by a factor of c. Another important part of this expression is that it is Lorentz invariant, meaning changing reference frames does not change the expression:

$$\sum_{\mu=0}^{3} p^\mu p_\mu \equiv -p_t^2 + p_x^2 + p_y^2 + p_z^2 = -c^2 m^2$$

14.2 Week 13: Day 2 (*11/20/25*)

4-momentum and the Schwarzschild Metric

We can take the Schwarzschild metric and divide by $d\tau$ to turn everything from momentum to velocity components:

$$-c^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t}^{\,2} + \frac{\dot{r}^{\,2}}{1 - \frac{2GM}{rc^2}} + r^2\dot{\theta}^{\,2} + r^2 \sin^2\theta\, \dot{\phi}^{\,2} \tag{2}$$

Then using our previous summation expression, we can substitute to find the 4-momentum. If we have our spherical coordinates (and time) in an orthonormal basis, this gives us:

$$p_t = mc\,\dot{t}, \qquad p_r = m\,\frac{\dot{r}}{\sqrt{1 - \frac{2GM}{rc^2}}}, \qquad p_\theta = mr\dot{\theta}, \qquad p_\phi = mr\sin\theta\,\dot{\phi}$$

And

$$\dot{t} = \frac{dt}{d\tau}$$

Which are useful expressions for the momentum of our particle when considering the Schwarzschild metric.

Variational Calculus Recap

While basic calculus can help us minimize the parameters of certain functions using derivatives, variational calculus can help us minimize or maximize integrals and functionals. If we have some function

$$f(x_i, \dot{x}_j) = p_1(x_1, \ldots, x_n)\dot{x}_1^2 + \cdots + p_n(x_1, \ldots, x_n)\dot{x}_n^2$$

And we want to minimize (or maximize) the integral:

$$\int_{\tau_1}^{\tau_2} d\tau\,\sqrt{f(x_i, \dot{x}_j)} \qquad \text{or} \qquad \int_{\tau_1}^{\tau_2} d\tau\, f(x_i, \dot{x}_j)$$

$$\int_{\tau_1}^{\tau_2} d\tau\, f(x_i, \dot{x}_j) = \int_{\tau_1}^{\tau_2} d\tau\, f(x_i + \delta x_i,\ \dot{x}_j + \delta\dot{x}_j)$$

We can Taylor expand this into the Euler-Lagrange equation (ELE). When there are multiple independent variables in the problem, there will be an ELE for each separate one.

$$\int_{\tau_1}^{\tau_2} d\tau \sum_{i=1}^{n} \left[\frac{\partial f}{\partial x_i}\delta x_i + \frac{\partial f}{\partial \dot{x}_i}\delta\dot{x}_i\right] = 0$$

If we then integrate this by parts:

$$\int_{\tau_1}^{\tau_2} d\tau \sum_{i=1}^{n} \left[\frac{\partial f}{\partial x_i} - \frac{d}{d\tau}\left(\frac{\partial f}{\partial \dot{x}_i} \right) \right] \delta x_i = 0$$

Thus the ELEs are:

$$\frac{\partial f}{\partial x_i} - \frac{d}{d\tau}\left(\frac{\partial f}{\partial \dot{x}_i} \right) = 0, \qquad i = 1, \ldots, n.$$

Where each index i represents a different variable in our original configuration.

Schwarzschild Geodesics

We can use variational calculus and the ELEs to find the geodesics on the Schwarzschild metric like we've done for other metrics in the past. If we want to extremize the metric, then we're considering the integral:

$$\int d\tau \left\{ -\left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{2GM}{rc^2}} + r^2 \dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2 \right\}$$

The ELEs become:

$$\frac{d}{d\tau}\left[\left(1 - \frac{2GM}{rc^2}\right) \dot{t} \right] = 0$$

Which can be written as:

$$\left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t} = \tilde{E} = \text{constant} \tag{1}$$

Which tells us that energy is conserved. If we take the orbit to lie in an equatorial plane, we can simplify our problem:

$$\theta = \frac{\pi}{2}$$

The equation for ϕ is then:

$$\frac{d}{d\tau}(r^2 \dot{\phi}) = 0$$

which can be written as

$$r^2 \dot{\phi} = \tilde{L} = \text{constant}$$

which gives us another conservation law, namely conservation of angular momentum. With these values for $\dot{t}$ and $\dot{\phi}$, we can plug in to find:

$$\dot{t} = \frac{\tilde{E}}{c^2 \left(1 - \frac{2GM}{rc^2}\right)}, \qquad \dot{\phi} = \frac{\tilde{L}}{r^2}$$

Which then gives us

$$\dot{r}^2 = c^2 E^2 - \left[1 - \frac{2GM}{rc^2}\right]\left(c^2 + \frac{\tilde{L}^2}{r^2}\right) = c^2\left[E^2 - \tilde{V}_{\text{eff}}^2(r)\right] \tag{3}$$

We can find what's called the 'effective potential':

$$\tilde{V}_{\text{eff}}^2(r) = c^2\left[1 - \frac{2GM}{rc^2}\right]\left[c^2 + \frac{\tilde{L}^2}{r^2}\right].$$

Circular orbits

So now we can see that the particle's energy is the same as the minimum of the potential it is subject to, a circular orbit. When the energy is larger than the minimum (for fixed $\tilde{L}$), the orbit becomes elliptical.

$$\dot{r} = 0 \qquad \rightarrow \qquad E = \tilde{V}_{\text{eff}}$$

The Effective Potential

We derived an expression for the effective potential using the ELEs.

Concept

The Effective Potential for a particle in the Schwarzschild Metric

$$\tilde{V}_{\text{eff}}^2(r) = c^2\left[1 - \frac{2GM}{rc^2}\right]\left[c^2 + \frac{\tilde{L}^2}{r^2}\right] \tag{3}$$

This effective potential has two different extrema points, which correspond to circular orbits. To find the extrema, we can use:

$$\frac{d\tilde{V}_{\text{eff}}}{dr} = 0$$

Which gives us:

$$R_s c^2 r^2 - 2r\tilde{L}^2 + 3R_s\tilde{L}^2 = 0, \qquad R_s = \frac{2GM}{c^2}$$

Using the quadratic formula:

$$\tilde{L}^2 = 3c^2 R_s^2, \qquad \text{or} \qquad \tilde{L} = \sqrt{12}\,M \quad (G = c = 1)$$

So stable circular orbits exist only for:

$$\tilde{L} > \sqrt{12}\,M$$

For the last stable circular orbit:

$$r = 6\,GM/c^2$$

This is how we know that a black hole can accrete 5.72% of the rest mass of gas into energy through radiation. Rotating black holes can convert much more, up to 20%.

Photon orbit around a Schwarzschild black hole

Now that we know how particles behave in the Schwarzschild metric, around black holes, we can consider two cases: For particles of non-zero rest-mass

$$-c^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t}^2 + \frac{r^2}{1 - \frac{2GM}{rc^2}} \dot{r}^2 + r^2 \dot{\phi}^2$$

And for photons ($ds^2 = 0$):

$$0 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t}^2 + \frac{r^2}{1 - \frac{2GM}{rc^2}} \dot{r}^2 + r^2 \dot{\phi}^2 \qquad \text{(for photons)}$$

The expressions we found earlier for $\dot{t}$ and $\dot{\phi}$ for particles are also valid for photons, and we can substitute to find:

$$\dot{r}^2 = c^{-2}\tilde{E}^2 - \left[1 - \frac{2GM}{rc^2}\right] \frac{\tilde{L}^2}{r^2} = c^{-2}\left[\tilde{E}^2 - \tilde{V}_{\text{eff}}^2\right]$$

Which rearranges to give as effective potential of:

$$\tilde{V}_{\text{eff}}^2(r) \equiv c^2 \left[1 - \frac{2GM}{rc^2}\right] \frac{\tilde{L}^2}{r^2}$$

Which is maximized at

$$r = \frac{3}{2} R_s = 3GM/c^2$$

This shows what we discussed earlier without justification: that photons have unstable circular orbits around this radius outside the event horizon, and will sometimes escape and sometimes fall in.

14.3 Week 13: Summary ()

The second-to-last week of the course was an extreme deep dive into black holes. We learned about some of their basic properties, observational evidence astronomers have uncovered to find them, and we spent most of the week discussing the dynamics of particles that are subject to their gravity.

15 Week 14

15.1 Week 14: Day 1 (*12/2/25*)

Stars falling into Supermassive black holes

Supermassive black holes that have mass larger than 10^6 M_{sun} have been recorded to 'eat' stars which wander too close to the black holes. If we consider the case in which a person hypothetically falls into a black hole, shearing tidal forces due to gravity causes more force to be exerted on the person's toes than their head. This same principle applies to a star which is distorted and sheared apart. As the black hole feeds on the star, star material streams toward the black hole and forms an accretioin disk. A particle jet known as a quasasr shoots out of the top of the black hole, containing about half of the energy that falls in. These quasars are extremely luminous X-ray and radio sources.

The tidal acceleration caused by the black hole is given by

$$a_T \approx GM_{BH}R_*/d^3$$

And the star's self-gravity is given by

$$a_* \approx GM_*/(R_*)^2$$

Knowing this, we can determine what happens when stars of different masses fall into black holes

Concept

Stars are tidally torn apart by black holes if $a_T > A_* \rightarrow d < R_T = R_*(M_{BH}/M_*)^{1/3}$
For a stellar mass star, tadal shredding occurs outside the event horizon for a BH smaller than $3 \times 10^7 M_{sun}$

Scaling for TDE Mass accretion

In the case where a star's trajectory brings it close to the black hole's tidal radius, we can calculate the rate at which mass is converted to luminosity around the black hole. To do this, recall the equations for energy in an orbit as well as Kepler's third law

Concept

Kepler's 3rd law: $P = 2\pi\sqrt{\frac{a^3}{GM_{BH}}} \rightarrow \propto \epsilon^{-3/2}$

Concept

Energy in an orbit: $\epsilon = -\frac{GM_{BH}}{2a} \rightarrow a \propto \epsilon^{-1}$

Then we can take the derivative of the period with respect to mass to find

$$\frac{dP}{dm} \propto \epsilon^{-5/2}$$

which is the proportionality of the mass fall-back rate.

Tidal Disruption Events

Tidal disruption events (TDEs) are events in which stars get close enough to a black hole to experience these tidal shearing effects. A very small amount of these events occur per year per galaxy, but 4 were discovered by ROSAT in the 90s as X-ray flares at $0.1keV$ in some galaxies. There are many other candidate observations of TDEs.

Radiation Generation

Relativistic jets of gas dissipate X-ray radiation, as we have discussed. Whenever we see a jet of powerful gas emitting in the X-ray, we are interested in examining its composition, and the acceleration of the particles within it. Generally, we expect either baryon or Poynting flux to be responsible for most of the energy.

Synchrotron Radiation

Synchrotron radiation is EM radiation that is emitted by electrons or other charged particles that are traveling through magnetic fields and changing direction. For a magnetic field with strength B' in a jet moving at bulk Lorentz factor Γ, the radiation frequency is:

$$\nu_i = \frac{qB'\gamma_i^2\Gamma}{2\pi m_e c(1+z)} \approx (1.2 \times 10^{-8}\text{ eV})\, B'\gamma_i^2\Gamma(1+z)^{-1}$$

Adding this up for N_e electrons, the total observed flux at that frequency is:

$$f_i = \frac{\sqrt{3}\, q^3\, B'\, N_e\, \Gamma(1+z)}{4\pi d_L^2\, m_e c^2}$$

$$\approx (1.8 \times 10^2\text{ mJy})\, N_{e,55}\, B'\, \Gamma\, (1+z)/d_{L,28}^2$$

This principle is how we can find the total amount of electrons that are emitted from the jet.

Hawking Radiation

Black holes don't last forever. Physicist Stephen Hawking predicted that radiation escapes black holes in the form of particle anti-particle pairs forming and being separated at the edges of the event horizon. The spectrum of black hole's radiation is very similar to a blackbody, and it has a temperature that is inversely proportional to itsmass.

Temperature of Hawking radiation

The temperature emitted by a black hole of mass M_{BH} is:

$$T_{\text{BH}}K \approx \frac{10^{26}}{M_{\text{BHg}}}$$

Peak wavelength of Hawking radiation

The peak wavelength of the blackbody spectrum for a black hole is

$$\lambda = \frac{b}{T}$$

where b Wien's constant. For Hawking radiation, the particle that escapes has a wavelength that is approximately the Schwarzschild radius $\lambda \sim R_s$, so

$$\lambda \sim R_s \quad \rightarrow \quad T_{\text{BH}} \sim \frac{b}{R_s}$$

And using the Schwarzschild radius

$$R_s = \frac{2GM_{\text{BH}}}{c^2}$$

we find that

$$T_{\text{BH}} \approx \frac{2 \times 10^{27}}{M_{\text{BH}}}$$

This is a simple estimate. This can then be written as

$$T_{\text{BH}} = \frac{\hbar c^3}{8\pi G M_{\text{BH}} k_B}$$

15.2 Week 14: Day 2 (*12/4/25*)

Quasars

We have informally introduced quasars already, but now we will take a dedicated look at them. Quasi stellar radio sources (Quasars) are the relativistic, high energy gas jets that erupt from super-massive black holes at the centers of galaxies that accrete gas. Infalling material is converted to energy in the form of radiation by the extreme gravity of the black holes. Quasars were discovered in 1963, when a Caltech astronomer identified an object that was brighter than all of the stars in its origin galaxy combined. The spectral features of quasars were very strange for the time, and they showed an extreme amount of redshift, which was seen to be a significant fraction of the speed of light. Nothing like this had been seen before, and it was thought to be the furthest object we were able to observe at the time due to this redshift.

Like we discussed previously, about half of the infalling material is 'eaten' by the black hole, and the other half is ejected out of the top of the black hole, because the infalling material must conserve angular momentum when it accretes.

Radio Galaxies

Many galaxies have active galactice nuclei (supermassive black holes) that have relativistic jets of material fly outward and colliding with the interstellar medium that is *kpc* away from the black hole, creating lobes around the nucleus. The energy of the jets that create these radial lobes is about the rest mass of 10 million stars. We can see these lobes emit in the X-ray.

Time Variability

Some quasars have brigthness variations on the timescale of weeks. Given that the luminosity can not vary faster than the light traveling to reach us, an object that has to take time to transition from bright to dark. We can use this property to determine, through the luminosity changing patterns, to determine what kind of object is emitting the light. For the above observations, we can deduce that only quasars (black holes) could produce light like this.

More on Gravitational Radiation

As we've discussed previously, gravitational waves are the gravitational analogue of waves propagating through water and shifting things they move through. They are distortions in the curvature of spacetime and propagate at the speed of light. In the 1970s, the orbits of two neutron stars in a binary orbit were measured. Gravitational waves propagating outward from the orbit contained energy, meaning that the orbit lost energy and the orbital period/radius started to decrease for the orbit over 20 years. By measuring the energies in their orbits and fitting it to models of energy loss from gravitational waves, we were able to find evidence for gravitational waves. Direct laboratory observation for gravitational waves was found in LIGO, as was previously mentioned.

15.3 Week 14: Summary ()

We finished off the course by focusing on supermassive black holes and their interactions with gas and stars, and how these interactions lead to quasars. Some interation phenomena are tidal disruption events in which material is tidally sheared off of stars into supermassive black holes, and radiation generation within this gas caused by synchrotron principles. Coming full circle, we discussed gravitational waves and their initial discovery.

16 Conclusion

Throughout the AST353 course, we have covered the important topics in modern astrophysics in both a qualitative and quantitative way. Beginning from the fundamental forces and building blocks of matter in our universe, we made our way to groundbreaking theoretical and experimental work confirming things like relativity and black holes, to observational techniques that allows us to probe the universe for its secrets. Thank you to Professor Kumar for the wonderful lectures and course experience, and to Eden for all of her help throughout the course.